WEST BENGAL STATE UNIVERSITY

B.Sc. Honours/Programme 3rd Semester Examination, 2020, held in 2021

MTMHGEC03T/MTMGCOR03T-MATHEMATICS (GE3/DSC3)

REAL ANALYSIS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
Candidates should answer in their own words and adhere to the word limit as practicable.
All symbols are of usual significance.

Answer Question No. 1 and any *five* from the rest

1. Answer any ***five*** questions from the following: $2\times5 = 10$

(a) Let $A \subseteq \mathbb{R}$ be a non empty set. When is A said to be bounded above? What do you mean by the least upper bound of A ?

(b) Give an example of a bounded above subset E of $\mathbb{R}$ for which sup E is not a limit point of E.

(c) Show that the sequence $\left\{\dfrac{3n+1}{n+1}\right\}$ is bounded.

(d) Examine the convergence of the sequence $\left\{\left(\dfrac{4}{5}\right)^n\right\}$.

(e) Show that the series $\sum\limits_{n=1}^{\infty} a_n$ converges, where

$$a_n = \frac{2n+3}{2n\,(n+1)\,(n+3)}\ ,\ \forall\, n \in \mathbb{N}$$

(f) Examine the convergence of the series $\sum\limits_{n=1}^{\infty} \dfrac{2^n n!}{n^n}$.

(g) Examine whether the sequence of functions $\{f_n\}$ converges uniformly on $\mathbb{R}$, where for all $n \in \mathbb{N}$,

$$f_n(x) = \frac{x^2}{n}\ ,\ x \in \mathbb{R}.$$

(h) Show that the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos x^2}{5n^6}$ is uniformly convergent on $\mathbb{R}$.

(i) Is $\sum_{n=1}^{\infty} 2^{-n}\cos(3^n x)$ a continuous function on $\mathbb{R}$? Justify your answer.

(j) Find the radius of convergence of the power series

$$x + \frac{(2!)^2}{4!}x^2 + \frac{(3!)^2}{6!}x^3 + \cdots\cdots$$

2. (a) Find the least upper bound and greatest lower bound of — 2+2

$$S = \{x \in \mathbb{R} \ : \ 3x^2 - 10x + 3 < 0\}$$

(b) Let S be a non empty bounded subset of $\mathbb{R}$ and let T be a non empty subset of S. Show T is a bounded subset of $\mathbb{R}$. Further show that — 2+2

$$\inf S \le \inf T \quad \text{and} \quad \sup T \le \sup S$$

3. (a) State and prove the Archimedean property of $\mathbb{R}$. — 1+3

(b) Find the least upper bound and greatest lower bound of — 2+2

$$S = \left\{ \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, -\frac{8}{7}, \ldots \right\}$$

4. (a) Justify that the set of integers $\mathbb{Z}$ has no cluster point. — 2

(b) Justify that a finite subset of $\mathbb{R}$ has no cluster point. — 2

(c) Show that 1 and –1 are limit points of the set — 4

$$S = \left\{ (-1)^m + \frac{1}{n} \ ; \ m, n \in \mathbb{N} \right\}$$

5. (a) Define a bijective map $f : \mathbb{N} \longrightarrow \mathbb{Z}$ to show that $\mathbb{Z}$ is countably infinite. Justify your answer. — 4

(b) Show that [0, 1] is an uncountable set. — 4

6. (a) Prove that limit of a convergent sequence is unique. — 4

(b) Let $x > 0$. Prove that $\lim_{n\to\infty} x^{1/n} = 1$. — 4

7. (a) Examine the monotonicity of the sequence $\{x_n\}$, where — 2+2+1

$$x_n = \frac{2n-1}{3n+4} \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence determine the convergence of $\{x_n\}$. If the sequence $\{x_n\}$ converges, find its limit.

(b) Use Cauchy's criterion for convergence to show that the sequence $\left\{\frac{n+1}{n}\right\}$ is convergent. 3

8. (a) Let $x \in \mathbb{R}$. Show that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$. 5

(b) Examine the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{n}{2n+1}\right)^n$. 3

9. (a) Show that if the series $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then $\sum_{k=1}^{\infty} a_k$ is convergent. Give an example, with justifications, to show that the converse may not be true. 4+2

(b) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$. 2

10.(a) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \frac{n^2x^2}{1+n^3x^3} \quad , \quad x \ge 0$$

is pointwise convergent on $[0, \infty)$ but is not uniformly convergent on $[0, \infty)$.

(b) Show that the sequence of functions $\{f_n\}$, where for all $n \in \mathbb{N}$, 4

$$f_n(x) = \begin{cases} nx & ; \quad 0 \le x \le \frac{1}{n} \\ 1 & ; \quad \frac{1}{n} < x \le 1 \end{cases}$$

is pointwise convergent on [0, 1] but is not uniformly convergent on [0, 1].

11.(a) Show that the series $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ is uniformly convergent on $\mathbb{R}$. 3

(b) Show that the series $\sum_{n=1}^{\infty} f_n(x)$ where 5

$$f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \quad , \quad x \in [0, 1]$$

is not uniformly convergent on [0, 1] but it can still be integrated term by term over [0, 1].

12.(a) Show that the series $\sum_{n=0}^{\infty}(1-x)x^n$ is not uniformly convergent on [0, 1]. 3

(b) Show that the series $\sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$ converges uniformly for all real values of x. Further, if $f(x)$ is the sum function of this series, then show that $f'(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$ for all $x\in\mathbb{R}$. 5

13.(a) Find the radius of convergence of the power series 3

$$\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n$$

(b) Use the fact that 5

$$\frac{1}{\sqrt{1-x^2}}=1+\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^{2n}\,,\ \forall\ |x|<1$$

to obtain the power series of $\sin^{-1}(x)$.

N.B. : *Students have to complete submission of their Answer Scripts through E-mail / Whatsapp to their own respective colleges on the same day / date of examination within 1 hour after end of exam. University / College authorities will not be held responsible for wrong submission (at in proper address). Students are strongly advised not to submit multiple copies of the same answer script.*

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